Journal of Applied Mechanics and Technical Physics, Vol. 46, No. 6, pp. 842–850, 2005

EFFECT OF RELAXATION AND RETARDATION TIME ON PERISTALTIC TRANSPORT OF THE OLDROYDIAN VISCOELASTIC FLUID

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UDC 532.536

The influence of relaxation and retardation time on peristaltic transport of an incompressible Oldroydian viscoelastic fluid by means of an infinite train of sinusoidal waves traveling along the walls of a two-dimensional flexible channel is investigated. A perturbation solution is obtained for the case in which the amplitude ratio (wave amplitude to channel half-width) is small. The results show that the values of the mean axial velocity of an Oldroydian viscoelastic fluid is smaller than that for a Newtonian fluid. The reflux phenomena are discussed. It is found that the critical reflux pressure gradient decreases with increasing retardation time and increases with increasing relaxation time. Numerical results are reported for different values of the physical parameters of interest.

Key words: *Oldroydian fluid, peristaltic motion.*

Introduction. The word "peristalsis" stems from the Greek word "peristalikos," which means clasping and compressing. Peristaltic transport of fluids occurs in the esophagus, the ureter, and the lower intestine. In addition, peristaltic pumping occurs in many practical applications involving biomechanical systems, such as roller and finger pumps. A mathematical analysis of peristaltic pumping in a two-dimensional formulation was presented by Latham [1]. Fung and Yih [2] investigated a perturbation solution of a two-dimensional case in which the amplitude ratio (wave amplitude to channel half-width) was small. Srivastava and Srivastava [3] studied the blood flow. El-Shehawey and Mekheimer [4] examined the effects of couple-stresses in peristaltic transport of fluid. Peristaltic transport of a particle–fluid suspension was considered in [5, 6]. Antanovskii and Ramkissoon [7] studied peristaltic transport of a compressible viscous fluid in a finite pipe. Carew and Pedley [8] investigated periodic activation waves in an infinite tube.

Most theoretical investigations were performed for Newtonian fluids, although it is known that most physiological fluids behave like non-Newtonian fluids. In this aspect, there is only limited information on transport of non-Newtonian fluids. The main reason is that additional nonlinear terms appear in equations of motion, rendering the problem more difficult to solve. Another reason is that a universal non-Newtonian constitutive relation that can be used for all fluids and flows is not available. The earliest studies date back to Raju and Devanathan [9, 10] who considered the motion of an inelastic power-law fluid and of a special differential-type viscoelastic fluid of grade two through a tube with sinusoidal small-amplitude corrugation in the axial direction. Bohme and Friedrich [11] investigated peristaltic flows of viscoelastic fluids under the assumptions that the relevant Reynolds number is small enough to neglect inertia forces and that the ratio of the wave length and the channel height is large, which implies that the pressure is constant over the cross section. Hayat et al. [12] investigated periodic unsteady flows of a non-Newtonian fluid. Misra and Pandey [13] studied a peristaltic flow of blood in small vessels by developing a mathematical model in which blood was treated as a two-layer fluid. Mernone et al. [14] considered a peristaltic flow of rheologically complex physiological fluids modeled by a non-Newtonian (Casson) fluid in a two-dimensional

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Ain Shams University, Roxy, Cairo, Egypt; aky333@yahoo.com. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 46, No. 6, pp. 86–95, November–December, 2005. Original article submitted November 18, 2004; revision submitted January 10, 2005.

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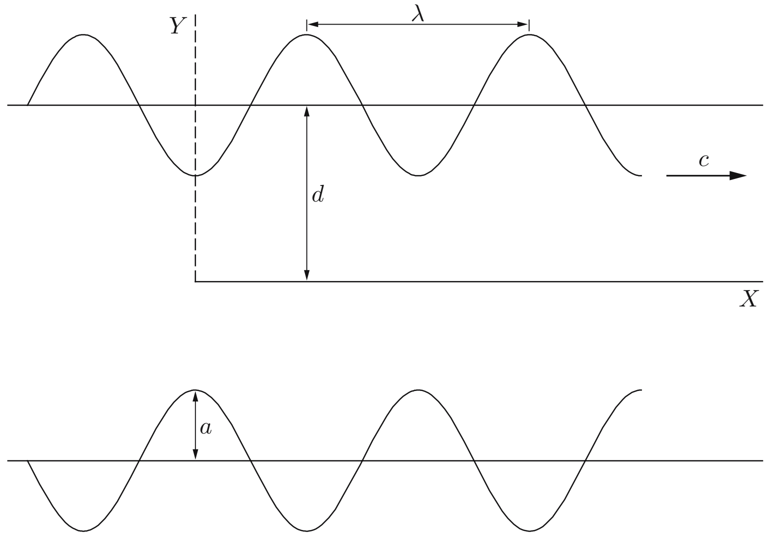

Fig. 1. Geometry of the problem.

channel. Hayat et al. [15] studied the effect of a third-order fluid on peristaltic transport in a circular cylindrical tube under the assumption that the wavelength of peristaltic waves is large compared to the mean tube radius. Hayat et al. [16] also considered a two-dimensional flow of a Johnson–Segalman fluid in a planar channel whose walls are transversely displaced by an infinite harmonic traveling wave of large wavelength.

The present paper considers peristaltic transport of an Oldroydian viscoelastic fluid with an arbitrary value of the Reynolds number. Such a work seems to be important and useful because attention has been hardly given to the Oldroydian fluid. Moreover, some non-Newtonian models take into account normal stress differences and shear thinning/thickening effects but lack other features, such as stress relaxation. In our analysis, we assume that the velocity components and the pressure gradient can be expanded in a regular perturbation series of the amplitude ratio. Nonlinearity of the equations of motion is taken into account. The combined effects of the relaxation time, retardation time, and material parameters of the fluid are examined. As the viscoelastic parameters tend to zero, the analytical results reduce to the well-known case of a Newtonian fluid and agree with the data obtained by Fung and Yih [2].

Basic Equations and Formulation of the Problem. We consider a two-dimensional channel of uniform width $2d$ filled by an incompressible Oldroydian viscoelastic fluid. We assume that an infinite wave train travels with a velocity c along the walls (Fig. 1). According to Oldroyd [17], the constitutive equations for the Oldroyd-B fluid are

$$\Sigma = -PI + S; \tag{1}$$

$$S + \lambda_1\Big(\frac{dS}{dt} - LS - SL^{\rm t}\Big) = \mu\Big(A_1 + \lambda_2\Big(\frac{dA_1}{dt} - LA_1 - A_1L^{\rm t}\Big)\Big), \tag{2}$$

where Σ is the Cauchy stress tensor, $-PI$ is the spherical part of stress due to incompressibility, d/dt is the material derivative with respect to time, μ is the viscosity, and λ_1 and λ_2 are material time constants referred to as the relaxation and retardation time, respectively. It is assumed that $\lambda_1 \geqslant \lambda_2 \geqslant 0$. The tensors L and A_1 are defined as follows:

$$L = \operatorname{grad} \boldsymbol{V}, \qquad A_1 = L + L^{\rm t}. \tag{3}$$

It should be noted that this model includes a viscous Navier–Stokes fluid as a special case for $\lambda_1 = \lambda_2 = 0$. Further, if $\lambda_2 = 0$, then it reduces to a Maxwellian fluid.

The equations of continuity and momentum for an incompressible fluid flow are given by

$$\operatorname{div} \boldsymbol{V} = 0; \tag{4}$$

$$\rho\,\frac{d\boldsymbol{V}}{dt} = \operatorname{div} \Sigma \tag{5}$$

(ρ is the density). The velocity field for an unsteady two-dimensional flow can be written as

$$\boldsymbol{V} = (u(x,y,t), v(x,y,t), 0). \tag{6}$$

From Eqs. (1)–(5) and (6), we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0;$$

$$\rho\Big(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\Big) = -\frac{\partial P}{\partial x} + \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y}; \tag{7}$$

$$\rho\Big(\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y}\Big) = -\frac{\partial P}{\partial y} + \frac{\partial S_{xy}}{\partial x} + \frac{\partial S_{yy}}{\partial y}; \tag{8}$$

$$S_{xx} + \lambda_1\Big(\frac{\partial S_{xx}}{\partial t} + u\,\frac{\partial S_{xx}}{\partial x} + v\,\frac{\partial S_{xx}}{\partial y} - 2S_{xx}\,\frac{\partial u}{\partial x} - 2S_{xy}\,\frac{\partial u}{\partial y}\Big)$$

$$= 2\mu\Big[\frac{\partial u}{\partial x} + \lambda_2\Big(\frac{\partial^2 u}{\partial t\,\partial x} + u\,\frac{\partial^2 u}{\partial x^2} + v\,\frac{\partial^2 u}{\partial x\,\partial y} - 2\Big(\frac{\partial u}{\partial x}\Big)^2 - \frac{\partial u}{\partial y}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)\Big)\Big]; \tag{9}$$

$$S_{xy} + \lambda_1\Big(\frac{\partial S_{xy}}{\partial t} + u\,\frac{\partial S_{xy}}{\partial x} + v\,\frac{\partial S_{xy}}{\partial y} - S_{xy}\,\frac{\partial u}{\partial x} - S_{yy}\,\frac{\partial u}{\partial y} - S_{xx}\,\frac{\partial v}{\partial x} - S_{xy}\,\frac{\partial v}{\partial y}\Big)$$

$$= \mu\Big[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \lambda_2\Big(\frac{\partial^2 u}{\partial t\,\partial y} + \frac{\partial^2 v}{\partial t\,\partial x} + \Big(u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big)\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) - 2\,\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y} - 2\,\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x}\Big)\Big]; \tag{10}$$

$$S_{yy} + \lambda_1\Big(\frac{\partial S_{yy}}{\partial t} + u\,\frac{\partial S_{yy}}{\partial x} + v\,\frac{\partial S_{yy}}{\partial y} - 2S_{xy}\,\frac{\partial v}{\partial x} - 2S_{yy}\,\frac{\partial v}{\partial y}\Big)$$

$$= 2\mu\Big[\frac{\partial v}{\partial y} + \lambda_2\Big(\frac{\partial^2 v}{\partial t\,\partial v} + u\,\frac{\partial^2 v}{\partial x\,\partial y} + v\,\frac{\partial^2 v}{\partial y^2} - 2\Big(\frac{\partial v}{\partial y}\Big)^2 - \frac{\partial v}{\partial x}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big)\Big)\Big]. \tag{11}$$

Let the vertical displacements of the upper and lower walls be η and $-\eta$, respectively. The geometry of the wall surface is defined as

$$\eta = a\cos\,(2\pi(x - ct)/\lambda), \tag{12}$$

where a is the amplitude, λ is the wavelength, and c is the wave velocity. The horizontal displacement is assumed to be zero. Hence, the boundary conditions for the fluid are

$$y = \pm d \pm \eta: \qquad u = 0, \quad v = \pm\frac{\partial \eta}{\partial t}. \tag{13}$$

We introduce the following dimensionless variables and parameters: $x^* = x/d$, $y^* = y/d$, $u^* = u/c$, $v^* = v/c$, $t^* = ct/d$, $p^* = p/(\rho c^2)$, $\eta^* = \eta/d$, $S^*_{xx} = dS_{xx}/(\mu c)$, $S^*_{xy} = dS_{xy}/(\mu c)$, $S^*_{yy} = dS_{yy}/(\mu c)$, amplitude ratio $\varepsilon = a/d$, wavenumber $\alpha = 2\pi d/\lambda$, Reynolds number $\mathrm{Re} = cd\rho/\mu$, and Weissenberg numbers $w_1 = c\lambda_1/d$ and $w_2 = c\lambda_2/d$.

In terms of the stream function $\psi(x,y,t)$, after eliminating P and dropping the asterisks over the symbols, Eqs. (7)–(13) become

$$\frac{\partial}{\partial t}\,\nabla^2\psi + \psi_y\nabla^2\psi_x - \psi_x\nabla^2\psi_y = \frac{1}{R}\,[S_{xx,xy} + S_{xy,yy} - S_{xy,xx} - S_{yy,yx}]; \tag{14}$$

$$S_{xx} + w_1[S_{xx,t} + \psi_y S_{xx,x} - \psi_x S_{xx,y} - 2\psi_{xy}S_{xx} - 2\psi_{yy}S_{xy}]$$

$$= 2[\psi_{xy} + w_2[\psi_{xyt} + \psi_y\psi_{yxx} - \psi_x\psi_{xyy} - 2\psi^2_{xy} - \psi_{yy}(\psi_{yy} - \psi_{xx})]]; \tag{15}$$

$$S_{xy} + w_1[S_{xy,t} + \psi_y S_{xy,x} - \psi_x S_{xy,y} - \psi_{yy}S_{yy} + \psi_{xx}S_{xx}]$$

$$= 2w_2\psi_{xy}\nabla^2\psi + \Big(1 + w_2\Big(\frac{\partial}{\partial t} + \psi_y,\frac{\partial}{\partial x} - \psi_x\,\frac{\partial}{\partial y}\Big)\Big)(\psi_{yy} - \psi_{xx}); \tag{16}$$

$$S_{yy} + w_1[S_{yy,t} + \psi_y S_{yy,x} - \psi_x S_{yy,y} + 2\psi_{xx} S_{xy} + 2\psi_{xy} S_{yy}]$$

$$= -2[\psi_{xy} + w_2[\psi_{xyt} + \psi_y \psi_{xxy} - \psi_x \psi_{xyy} + 2\psi_{xy}^2 - \psi_{xx}(\psi_{yy} - \psi_{xx})]]; \tag{17}$$

$$\eta = \varepsilon \cos(\alpha(x - t));$$

$$y = \pm 1 \pm \eta: \qquad \psi_y = 0, \qquad \psi_x = \mp \alpha\varepsilon \sin(\alpha(x - t)), \tag{18}$$

where ∇^2 denotes the Laplacian operator and the subscripts indicate partial differentiation.

Method of Solution. We obtain the solution for the stream function as a power series in terms of the small parameter ε by expanding ψ, S_{xx}, S_{xy}, S_{yy}, and dp/dx in the following form:

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \ldots ; \tag{19}$$

$$\Big(\frac{\partial p}{\partial x}\Big) = \Big(\frac{\partial p}{\partial x}\Big)_0 + \varepsilon\Big(\frac{\partial p}{\partial x}\Big)_1 + \varepsilon^2\Big(\frac{\partial p}{\partial x}\Big)_2 + \ldots ; \tag{20}$$

$$S_{xx} = S_{xx0} + \varepsilon S_{xx1} + \varepsilon^2 S_{xx2} + \ldots ; \tag{21}$$

$$S_{xy} = S_{xy0} + \varepsilon S_{xy1} + \varepsilon^2 S_{xy2} + \ldots ; \tag{22}$$

$$S_{yy} = S_{yy0} + \varepsilon S_{yy1} + \varepsilon^2 S_{yy2} + \ldots . \tag{23}$$

The first term on the right side in Eq. (20) corresponds to the imposed pressure gradient associated with the primary flow, and the other terms correspond to peristaltic motion. Substituting Eqs. (19)–(23) into Eqs. (14)–(17) and (18) and collecting terms of like powers of ε, we obtain three sets of coupled differential equations with their corresponding boundary conditions in ε_0, ε_1, and ε_2. The first set of differential equations in ε_0, subject to a steady parallel flow and transverse symmetry assumption for a constant pressure gradient in the x direction, yields

$$\psi_0 = K\Big[y - \frac{y^3}{3}\Big], \qquad K = -\frac{R}{2}\Big(\frac{\partial P}{\partial x}\Big)_0. \tag{24}$$

The last solution (24) agrees with the results of Fung and Yih [2], which means that the flow at this order is independent of viscoelastic parameters. The second and third sets of differential equations in ψ_1 and ψ_2 with their corresponding boundary conditions reduce to the following relations:

$$\psi_1(x, y, t) = (\varphi_1(y)\,\mathrm{e}^{i\alpha(x-t)} + \varphi_1^*(y)\,\mathrm{e}^{-i\alpha(x-t)})/2; \tag{25}$$

$$S_{xx1}(x, y, t) = (\varphi_2(y)\,\mathrm{e}^{i\alpha(x-t)} + \varphi_2^*(y)\,\mathrm{e}^{-i\alpha(x-t)})/2; \tag{26}$$

$$S_{xy1}(x, y, t) = (\varphi_3(y)\,\mathrm{e}^{i\alpha(x-t)} + \varphi_3^*(y)\,\mathrm{e}^{-i\alpha(x-t)})/2; \tag{27}$$

$$S_{yy1}(x, y, t) = (\varphi_4(y)\,\mathrm{e}^{i\alpha(x-t)} + \varphi_4^*(y)\,\mathrm{e}^{-i\alpha(x-t)})/2; \tag{28}$$

$$\psi_2(x, y, t) = (\varphi_{20}(y) + \varphi_{22}(y)\,\mathrm{e}^{2i\alpha(x-t)} + \varphi_{22}^*(y)\,\mathrm{e}^{-2i\alpha(x-t)})/2; \tag{29}$$

$$S_{xx2}(x, y, t) = (\varphi_{30}(y) + \varphi_{33}(y)\,\mathrm{e}^{2i\alpha(x-t)} + \varphi_{33}^*(y)\,\mathrm{e}^{-2i\alpha(x-t)})/2; \tag{30}$$

$$S_{xy2}(x, y, t) = (\varphi_{40}(y) + \varphi_{44}(y)\,\mathrm{e}^{2i\alpha(x-t)} + \varphi_{44}^*(y)\,\mathrm{e}^{-2i\alpha(x-t)})/2; \tag{31}$$

$$S_{yy2}(x, y, t) = (\varphi_{50}(y) + \varphi_{55}(y)\,\mathrm{e}^{2i\alpha(x-t)} + \varphi_{55}^*(y)\,\mathrm{e}^{-2i\alpha(x-t)})/2 \tag{32}$$

(the asterisk denotes complex conjugation). Substituting Eqs. (25)–(32) into the differential equations and their corresponding boundary conditions in ψ_1 and ψ_2, we obtain three sets of coupled linear differential equations with their corresponding boundary conditions. These equations are sufficient to determine the solution up to the second order in ε. These equations, however, are fourth-order ordinary differential equations with variable coefficients, the boundary conditions are not all homogeneous, and the problem is not an eigenvalue problem. Nevertheless, we can restrict our investigation to the case of free pumping. Physically, this means that the fluid is stationary if there are no peristaltic waves. In this case, we put $(dp/dx)_0 = 0$, which means that $K = 0$; under this assumption, we obtain

$$i\alpha R(\varphi_1'' - \alpha^2\varphi_1) = i\alpha\varphi_4' - i\alpha\varphi_2' - \varphi_3'' - \alpha^2\varphi_3; \tag{33}$$

$$(1 - i\alpha w_1)\varphi_2 = 2i\alpha(1 - i\alpha w_2)\varphi_1'; \tag{34}$$

$$(1 - i\alpha w_1)\varphi_3 - \alpha^2(1 + i\alpha w_2)\varphi_1 = (1 - i\alpha w_2)\varphi_1''; \tag{35}$$

$$(1 - i\alpha w_1)\varphi_4 = -2i\alpha(1 - i\alpha w_2)\varphi_1', \tag{36}$$

where

$$\varphi_1(\pm 1) = \pm 1, \qquad \varphi_1'(\pm 1) = 0$$

and

$$\varphi_{40}'' = i\alpha R(\varphi_1^*\varphi_1'' - \varphi_1\varphi_1^{*\prime\prime})'/2; \tag{37}$$

$$\begin{aligned}\varphi_{30} = {} & -i\alpha w_1(\varphi_1^*\varphi_2 - \varphi_1\varphi_2^*)'/2 - 2w_2(\alpha^2(\varphi_1\varphi_1^{*\prime\prime} + \varphi_1''\varphi_1^*) + \varphi_1''\varphi_1^{*\prime\prime} + 3\alpha^2\varphi_1'\varphi_1^{*\prime}) \\ & - w_1(\varphi_3\varphi_1^{*\prime\prime} + \varphi_3^*\varphi_1'') + i\alpha w_1(\varphi_1'\varphi_2^* - \varphi_2\varphi_1^{*\prime});\end{aligned} \tag{38}$$

$$\begin{aligned}\varphi_{40} = {} & w_1(\varphi_1''\varphi_4^* + \varphi_1^{*\prime\prime}\varphi_4)'/2 + \alpha^2 w_1(\varphi_1\varphi_2^* + \varphi_2\varphi_1^*)/2 + i\alpha^3 w_2(\varphi_1\varphi_1^{*\prime} - \varphi_1'\varphi_1^*) + \varphi_{20}'' \\ & - i\alpha w_2(\varphi_1\varphi_1^{*\prime\prime} - \varphi_1^*\varphi_1'')'/2 - i\alpha w_1(\varphi_3\varphi_1^{*\prime} - \varphi_1'\varphi_3^*)'/2;\end{aligned} \tag{39}$$

$$\begin{aligned}\varphi_{50} = {} & i\alpha w_1(\varphi_4\varphi_1^* - \varphi_4^*\varphi_1)/2 + \alpha^2 w_2(\varphi_1\varphi_1^{*\prime} + \varphi_1^*\varphi_1')' + \alpha^2 w_1(\varphi_1\varphi_3^* - \varphi_1^*\varphi_3) \\ & - 4\alpha^2 w_2\varphi_1'\varphi_1^{*\prime} - \alpha^2 w_2(\varphi_1\varphi_1^{*\prime\prime} + \varphi_1^*\varphi_1'' + 2\alpha^2\varphi_1\varphi_1^*),\end{aligned} \tag{40}$$

where

$$\varphi_{20}'(\pm 1) = \mp(\varphi_1''(\pm 1) + \varphi_1^{*\prime\prime}(\pm 1))/2, \tag{41}$$

and

$$4\alpha R(\varphi_{22}'' - 4\alpha^2\varphi_{22}) = \alpha R\varphi_1'\varphi_1'' - \alpha R\varphi_1\varphi_1''' - 4\alpha\varphi_{33}' + 2i\varphi_{44}'' + 8i\alpha^2\varphi_{44} + 4\alpha\varphi_{55}',$$

$$(1 - 2i\alpha w_1)\varphi_{33} = i\alpha w_1(\varphi_1\varphi_2' + \varphi_1'\varphi_2)/2 - w_1\varphi_3\varphi_1'' + \alpha^2 w_2\varphi_1'^2 - w_2\varphi_1''^2 + 4i\alpha(1 - 2i\alpha w_2)\varphi_{22}',$$

$$\begin{aligned}(1 - 2i\alpha w_1)\varphi_{44} = {} & -i\alpha w_2(\varphi_1\varphi_1''' - 3\varphi_1'\varphi_1'')/2 - i\alpha^3 w_2\varphi_1\varphi_1' - 4i\alpha(i\alpha + 2\alpha^2 w_2)\varphi_{22} \\ & + i\alpha w_1(\varphi_1\varphi_3' - \varphi_1'\varphi_3)/2 + w_1(\varphi_1''\varphi_4 + \alpha^2\varphi_1\varphi_2)/2 + (1 - 2i\alpha w_2)\varphi_{22}'',\end{aligned}$$

$$\begin{aligned}(1 - 2i\alpha w_1)\varphi_{55} = {} & i\alpha w_1(\varphi_1\varphi_4' - 3\varphi_1'\varphi_4)/2 + \alpha^2 w_1\varphi_1\varphi_3 - 4i\alpha\varphi_{22}' + 3\alpha^2 w_2\varphi_1'^2 \\ & - 2\alpha^2 w_2\varphi_1\varphi_1'' - \alpha^4 w_2\varphi_1^2 - 8\alpha^2 w_2\varphi_{22}',\end{aligned}$$

where

$$\varphi_{22}(\pm 1) = \mp\varphi_1'(\pm 1)/4, \qquad \varphi_{22}'(\pm 1) = \mp\varphi_1''(\pm 1)/2.$$

The prime here denotes the derivative with respect to y. The solutions of Eqs. (33)–(36) are

$$\varphi_1(y) = A_1\sinh(\alpha y) + B_1\sinh(\beta y), \qquad \varphi_2(y) = A_2\cosh(\alpha y) + B_2\cosh(\beta y),$$

$$\varphi_3(y) = A_3\sinh(\alpha y) + B_3\sinh(\beta y), \qquad \varphi_4(y) = -A_2\cosh(\alpha y) - B_2\cosh(\beta y),$$

where

$$A_1 = -\beta\cosh\beta/(\alpha\cosh\alpha\sinh\beta - \beta\cosh\beta\sinh\alpha), \qquad B_1 = \alpha\cosh\alpha/(\alpha\cosh\alpha\sinh\beta - \beta\cosh\beta\sinh\alpha),$$

$$A_2 = 2i\alpha^2\Gamma A_1, \qquad B_2 = 2i\alpha\beta\Gamma B_1,$$

$$A_3 2\alpha^2\Gamma A_1, \qquad B_3 = \Gamma(\alpha^2 + \beta^2)B_1,$$

$$\beta^2 = \alpha^2 - i\alpha R/\Gamma, \qquad \Gamma = (1 + \alpha^2 w_1 w_2 + i\alpha(w_1 - w_2))/(1 + \alpha^2 w_1^2).$$

Next, in the expansion of ψ_2, we need only concern ourselves with the terms $\varphi'_{20}(y)$, because our aim is to determine the mean flow only. Thus, the differential equations (37)–(40) subject to the boundary condition (41) yield the expression

$$\varphi'_{20}(y) = F(y) - F(1) + D - C_1(1 - y^2),$$

where

$$D = \varphi'_{20}(\pm 1) = -[\alpha^2(A_1 + A_1^*)\sinh\alpha + \beta^2 B_1 \sinh\beta + \beta^{*2} B_1^* \sinh\beta^*]/2,$$

$$F(y) = s_1 \cosh((\alpha+\beta^*)y) + s_2 \cosh((\alpha-\beta^*)y) + s_3 \cosh((\alpha+\beta)y) + s_4 \cosh((\alpha-\beta)y)$$

$$+ s_5 \cosh((\beta+\beta^*)y) + s_6 \cosh((\beta-\beta^*)y) + s_7 \cosh(2\alpha y),$$

$$s_1 = i\alpha R(\alpha-\beta^*)A_1B_1^*/(4(\alpha+\beta^*)) - i\alpha w_1(A_1B_3^* - A_3B_1^*)/4$$

$$+ w_1(\beta^*-\alpha)A_2B_1^*/4 + i\alpha w_2(\beta^*-\alpha)^2 A_1B_1^*/4,$$

$$s_2 = -i\alpha R(\alpha+\beta^*)A_1B_1^*/(4(\alpha-\beta^*)) + i\alpha w_1(A_1B_3^* - A_3B_1^*)/4$$

$$+ w_1(\beta^*+\alpha)A_2B_1^*/4 - i\alpha w_2(\beta^*+\alpha)^2 A_1B_1^*/4,$$

$$s_3 = -i\alpha R(\alpha-\beta)A_1B_1^*/(4(\alpha+\beta)) + i\alpha w_1(B_3A_1^* - B_1A_3^*)/4$$

$$+ w_1(\beta-\alpha)B_1A_2^*/4 - i\alpha w_2(\beta-\alpha)^2 B_1A_1^*/4,$$

$$s_4 = i\alpha R(\alpha+\beta)A_1B_1^*/(4(\alpha-\beta)) + i\alpha w_1(B_1A_3^* - B_3A_1^*)/4$$

$$+ w_1(\beta+\alpha)B_1A_2^*/4 - i\alpha w_2(\beta+\alpha)^2 B_1A_1^*/4,$$

$$s_5 = -i\alpha R(\beta^*-\beta)B_1B_1^*/(4(\beta^*+\beta)) + i\alpha w_1(B_3B_1^* - B_1B_3^*)/4 + w_1(\beta^2-\alpha^2)B_1B_2^*/(4(\beta^*+\beta))$$

$$+ w_1(\beta^{*2}-\alpha^2)B_2B_1^*/(4(\beta^*+\beta)) + i\alpha w_2(\beta-\beta^*)(2\alpha^2-\beta^2-\beta^{*2})B_1B_1^*/(4(\beta^*+\beta)),$$

$$s_6 = i\alpha R(\beta^*+\beta)B_1B_1^*/(4(\beta^*-\beta)) - i\alpha w_1(B_3B_1^* - B_1B_3^*)/4 + w_1(\beta^2-\alpha^2)B_1B_2^*/(4(\beta^*-\beta))$$

$$+ w_1(\beta^{*2}-\alpha^2)B_2B_1^*/(4(\beta^*-\beta)) + i\alpha w_2(\beta+\beta^*)(2\alpha^2-\beta^2-\beta^{*2})B_1B_1^*/(4(\beta^*-\beta)),$$

$$s_7 = i\alpha w_1(A_3A_1^* - A_1A_3^*)/4.$$

Thus, we see that one constant C_1 remains arbitrary in the solution. Substituting Eqs. (19)–(23) into (7) under the assumption that $K = 0$, we find that

$$C_1 = R\Big(\frac{\partial\bar{p}}{\partial x}\Big)_2.$$

This also means that the time-averaged velocity can be written as

$$\bar{u}(y) = \frac{\varepsilon^2}{2}\,\varphi'_{20}(y) = \frac{\varepsilon^2}{2}\Big(F(y) - F(1) + D - R\Big(\frac{\partial\bar{p}}{\partial x}\Big)_2(1-y^2)\Big). \tag{42}$$

Note, if we put the Weissenberg numbers w_1 and w_2 equal to zero, then the results of the problem reduce to the solution found in [2] for a Newtonian fluid.

Numerical Results and Discussion. The problem of peristaltic motion of an Oldroydian viscoelastic fluid is controlled by viscoelastic parameters, wavenumber, Reynolds number, and second-order time-averaged pressure gradient. In this Section, the mean velocity at the channel boundaries, mean-velocity perturbation function, time-averaged mean axial-velocity distribution, and reflux are calculated for different values of these parameters in the free-pumping case. Numerical calculations based on Eq. (42) show that the mean axial velocity of the fluid due to peristaltic motion is dominated by the constant D, parabolic term $-R(\partial\bar{p}/\partial x)_2(1-y^2)$, and perturbation term $F(y) - F(1)$. The constant D, which initially arose from the non-slip condition of the axial velocity on the

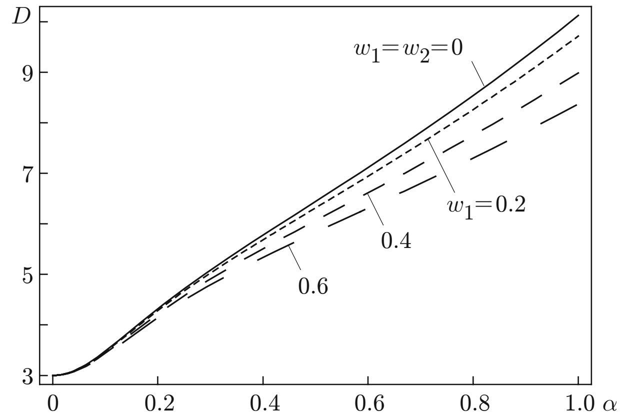

Fig. 2

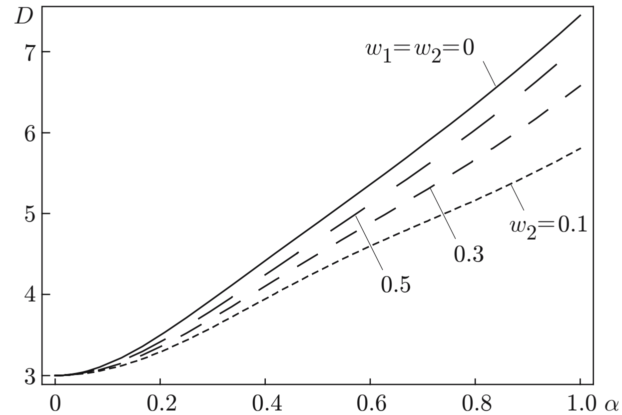

Fig. 3

Fig. 2. Effect of the viscoelastic parameter w_1 on the dependence $D(\alpha)$ for $w_2 = 0.1$ and $R = 100$.

Fig. 3. Effect of the viscoelastic parameter w_2 on the dependence $D(\alpha)$ for $w_1 = 0.8$ and $R = 50$.

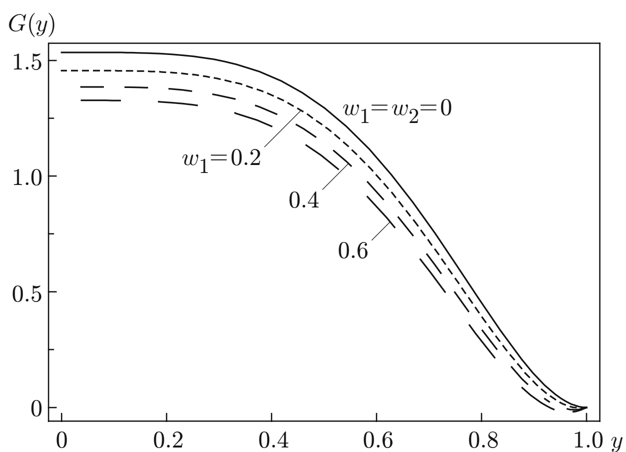

Fig. 4

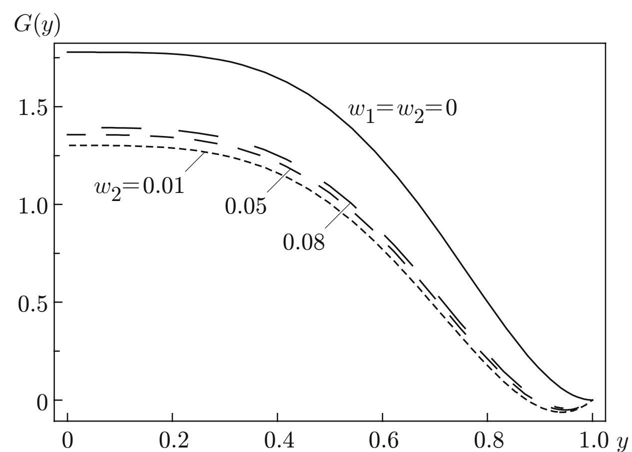

Fig. 5

Fig. 4. Effect of the viscoelastic parameter w_1 on the mean-velocity perturbation function $G(y)$ for $w_2 = 0.1$, $w_2 = 0.01$, and $R = 100$.

Fig. 5. Effect of the viscoelastic parameter w_2 on the mean-velocity perturbation function $G(y)$ for $w_2 = 0.1$, $w_1 = 0.5$, and $R = 50$.

wall, is due to the value of φ'_{20} at the boundary and is related to the mean velocity at the channel boundaries as $\bar{u}(\pm) = \varepsilon^2\varphi'_{20}(\pm)/2 = \varepsilon^2 D/2$. The parabolic term $-R(\partial\bar{p}/\partial x)_2(1-y^2)$ is negative for a positive pressure gradient and vice versa. The perturbation term $F(y) - F(1)$ is negative and proportional to α^2R^2. We define the mean-velocity perturbation function $G(y)$, as in [2], in the form

$$G(y) = -200(F(y) - F(1))/(\alpha^2R^2),$$

which yields

$$\bar{u}(y) = \frac{\varepsilon^2}{2}\Big(D - R\Big(\frac{\partial\bar{p}}{\partial x}\Big)_2(1-y^2) - \frac{\alpha^2R^2}{200}\,G(y)\Big). \tag{43}$$

Figures 2 and 3 show the dependence $D(\alpha)$ for different values of the Weissenberg numbers w_1 and w_2. The numerical results indicate that D decreases with increasing w_1 and increases with increasing w_2 and α. The dependence $G(y)$ for different values of w_1 and w_2 is plotted in Figs. 4 and 5. The results reveal that G decreases with increasing w_1 and increases with increasing w_2. The flow reflux will occur whenever there is a negative mean

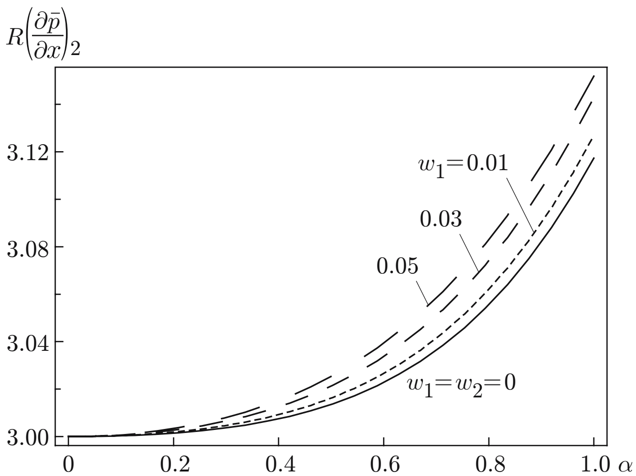

Fig. 6

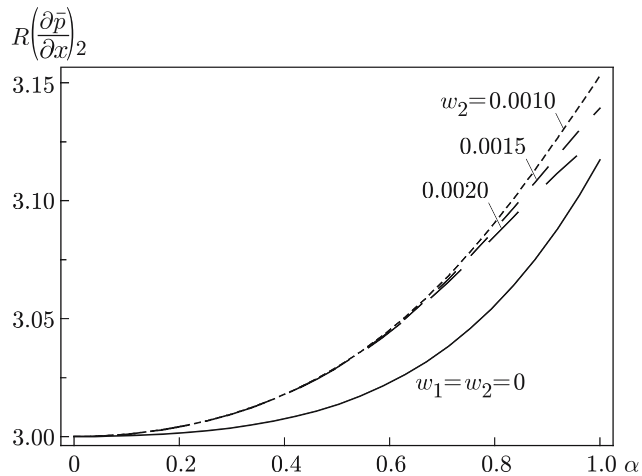

Fig. 7

Fig. 6. Effect of the viscoelastic parameter w_1 on variation of the critical reflux pressure gradient versus the wavenumber α for $w_2 = 0.001$ and $R = 10$.

Fig. 7. Effect of the viscoelastic parameter w_2 on variation of the critical reflux pressure gradient versus the wavenumber α for $w_1 = 0.08$ and $R = 10$.

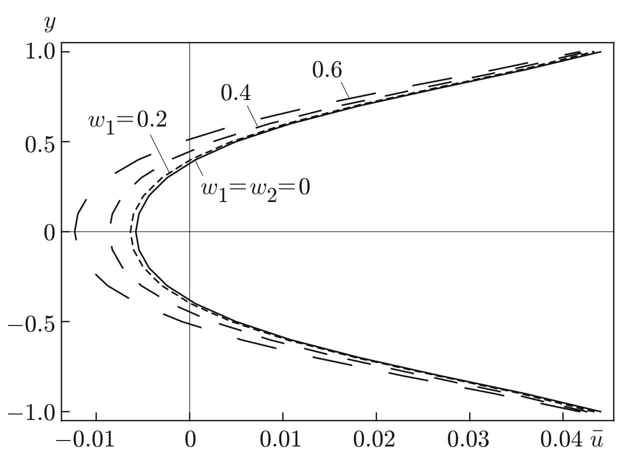

Fig. 8

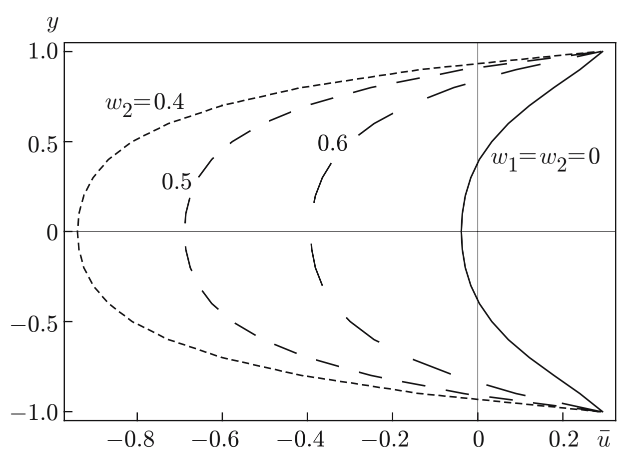

Fig. 9

Fig. 8. Effect of the viscoelastic parameter w_1 on the mean-velocity distribution and reflux for $(\partial\bar{p}/\partial x)_2 = 0.04$, $w_2 = 0.01$, $\alpha = 0.2$, $\varepsilon = 0.15$, and $R = 75$.

Fig. 9. Effect of the viscoelastic parameter w_2 on the mean-flow distribution and reflux for $(\partial\bar{p}/\partial x)_2 = 0.04$, $w_1 = 0.8$, $\alpha = 0.2$, $\varepsilon = 0.15$, and $R = 75$.

velocity in the flow field. Since $\bar{u}(\pm) = \varepsilon^2 D/2$, and D is always positive in the pure peristalsis case, the reflux will never occur at the boundaries. Furthermore, it follows from Eq. (43) that, if we put $\bar{u}(y) = 0$ on the centerline $y = 0$, then the critical reflux condition is

$$\Big(\frac{\partial \bar{p}}{\partial x}\Big)_{2,\mathrm{cr}} = \frac{1}{R}\Big(D - \frac{\alpha^2 R^2}{200}\, G(0)\Big).$$

The reflux occurs when $(\partial\bar{p}/\partial x)_2 > (\partial\bar{p}/\partial x)_{2,\mathrm{cr}}$. Figures 6 and 7 show the variation of $(\partial\bar{p}/\partial x)_{2,\mathrm{cr}}$ versus α for different values of w_1 and w_2. The calculation results reveal that $(\partial\bar{p}/\partial x)_{2,\mathrm{cr}}$ decreases with increasing w_2 and increases with increasing w_1. The zeroth-order solution is found to be identical to that for the Newtonian behavior. At this order, the Weissenberg numbers only contribute to S_{xx0}. Higher-order solutions were studied to reveal the effect of the non-Newtonian behavior on peristaltic waves. The results indicate that the second-order solution depends strongly on the Weissenberg numbers. The effect of the Weissenberg numbers w_1 and w_2 on the mean velocity and reflux is displayed in Figs. 8 and 9. We see that the reflux velocity increases with increasing w_1 and decreases with increasing w_2.

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